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A Fourth Order One-Step Hybrid Method for the Numerical Solution of Initial Value Problems of Second Order Ordinary Differential Equations

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Abstract

Our focus in this paper is the development and implementation of a new one-step hybrid method for direct solution of general second order ordinary differential equations. We adopted the Stomer cowell approach, in the development of the method, by taking the Power series as the basis function, from where the arising differential systems are collocated at all the selected grid and off grid points while the approximate solutions are interpolated at two points nearest to the end points. We then analyzed the resulting scheme for its basic properties, order of accuracy, error constant, consistency, zero stability and convergence. The method was found to have high order of accuracy, minimal error constant, Zero stable, consistent and convergent.

For the implementation of the method, we adopted Taylor's series algorithm to supply the starting values while the efficiency of the method is then tested on some sample problems and its accuracy compared with the existing methods. The method was found to perform better than the existing method.

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Introduction

Differential equations are the basis for modeling varieties of problems arising from the fields of sciences, technology, engineering, social science, e.t.c. These problems range from first order to higher order problems like falling body problems, orthogonal trajectories, damped mechanical oscillator, electrical circuit problems, and so on.

In this work, we consider a general second order initial value problem (IVP) of the form:

$$y'' = f(x, y, y'), y(a) = y_0, y'(a) = \eta \qquad (1.1)$$

Conventionally, problems of the form (1.1) are solved by first reducing it to systems of first order ordinary differential equations of the form:

$$y' = f(x, y), y(x_0) = y_0, \; x, y \in R^n, f \in C'[a, b] \qquad (1.2)$$

The resulting equations are then solved by any appropriate numerical or analytical methods. This is widely discussed by Fatunla (1988), Awoyemi (1992), Lambert (1973,1991), Lambert & Watson (1976), among others. This approach is considered uneconomical as a result of computational burdens, human and computer time wastage [Awoyemi & Kayode (2002)].

Consequently, many eminent scholars have developed other methods of solutions of (1.1) without first reducing it to systems of first order ODES. Among these are: Awoyemi (1999, 2001), Awoyemi & Kayode (2005), Yahaya (2009), Kayode (2004, 2005), Onumanyi, Sirisena & Jator (1999) to mention but just a few. These researchers proposed Linear Multistep Methods (LMM) with continuous coefficients for IVPs of the form (1.1) in the predictor-corrector mode based on collocation method using power series polynomial as the basis function and Taylor's series algorithm to supply starting values.

Another interesting approach to the solution of higher order ordinary differential equation is Hybrid method which has been developed by many scholars. Notable among these researchers are: Fatunla (1984) who developed a P-stable, one-leg hybrid LMM in which Pade approximation was used as a basis function for solving (1.1); Awoyemi (1995) proposed a two-step hybrid multistep methods with continuous coefficients for the solution of (1.2). Other scholars who have developed hybrid method include: Onumanyi et al (2001), Kayode (2007,2011), Yahaya & Badmus (2009), and Bolarinwa Bolaji (2012) etc.

Furthermore, block methods have been proposed by eminent scholars such as Fatunla (1991, 1994), for the solution of special second order ODEs. Yahaya (2007), Awoyemi, Adesanya & Ogunyebi (2009), Adesanya et al (2008, 2009), used the forward

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difference method and Newton's polynomial respectively to generate predictors for Fatunla's block method in order to solve equation (1.1). Lately, scholars like Jator (2007), Jator & Li (2009) have proposed five-step and four-step starting methods which adopt continuous LMM to obtain finite difference method applied as a block for the direct solution of (1.1). Adesanya (2011) adopted a method of collocation and interpolation to develop a continuous LMM which is evaluated at different grid points to give discrete methods and then adopted block methods approach in the discrete methods to generate independent solutions. Anake (2011) proposed hybrid one-step block methods that can solve (1.1) directly.
It should however be noted that little has been said about one-step hybrid methods with the use of Taylor's series algorithm as the major method of implementation. Consequently this work is aimed at developing one-step hybrid Stomer Cowell type method whose mode of implementation is the Taylor's series algorithm mode to solve directly equation (1.1).

Derivation of the Method

We proposed an approximate solution to the Initial Value Problems of the kind (1.1) in the form:

$$y(x)=\sum_{j=0}^{(m+n)-1} a_j x^j \qquad (2.1)$$

Which is a power series with a single variable x and (m+n) is the sum of the collocation and interpolation points.
The first and second derivatives of (2.1) are respectively:

$$y'(x)=\sum_{j=1}^{(m+n)-1} j a_j x^{j-1} \qquad (2.2)$$

and

$$y''(x)=\sum_{j=2}^{(m+n)-1} j(j-1) a_j x^{j-2} \qquad (2.3)$$

By combining (1.1) and (2.3), we have the differential system:

$$\sum_{j=2}^{(m+n)-1} j(j-1) a_j x^{j-2} = f(x,y,y') \qquad (2.4)$$

By collocating (2.4) at selected grid and off grid points, $x = x_{n+i}$, $0 \le i \le 1$ and interpolating (2.1) at selected grid and off grid points for the method result into a system of equations:

$$\sum_{j=2}^{(m+n)-1} j(j-1) a_j x^{j-2} = f_{n+i}, 0 \le i \le 1 \qquad (2.5)$$

And $$\sum_{j=0}^{(m+n)-1} a_j x^j = y_{n+i}\ ,\ 0 \le i \le 1 \qquad (2.6)$$

Where $$x_{n+i} = x_n + ih \qquad (2.7)$$

Which when solved for a_j'S yield a method expressed in the form:

$$y(x)=\sum_{j=0}^{k}\alpha_j(x) y_{n+j} + \sum_{j=0}^{k}\beta_j(x) f_{n+j} \qquad (2.8)$$

Where $k=1$ and $f_{n+j} = f\left(x_{n+j}, y_{n+j}, y^1_{n+j}\right)$, $0 \le j \le 1$

The coefficients $\alpha_j(x)$ and $\beta_j(x)$ in Equation (2.8) are as follows if

$$t=\frac{x-x_{n+\frac{2}{3}}}{h} \qquad (2.9)$$

$$\alpha_{\frac{1}{3}}(t) = -3t \qquad \alpha_{\frac{2}{3}}(t) = 1+3t$$

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$$\beta_0(t)=\frac{h^2}{1080}\left(-7t+90t^3-243t^5\right),\ \beta_{\frac{1}{3}}(t)=\frac{h^2}{1080}\left(66t-540t^3+405t^4+729t^5\right)$$

$$\beta_{\frac{2}{3}}(t)=\frac{h^2}{1080}\left(129t+540t^2+270t^3-810t^4-729t^5\right),\ \beta_1(t)=\frac{h^2}{1080}\left(-8t+180t^3+405t^4+243t^5\right)$$

(2.10)

$$y_{n+1}=2y_{n+\frac{2}{3}}-y_{n+\frac{1}{3}}+\frac{h^2}{108}\left(f_{n+1}+10f_{n+\frac{2}{3}}+f_{n+\frac{1}{3}}\right) \tag{2.11}$$

with the order $C_4=4$, error constant $C_6=\frac{-1}{174960}$ and interval of absolute stability

$$X(\theta)=\left(-9.8,0\right).$$

The first derivative is given by:

$$y'_{n+1}=\frac{1}{h}\left(3y_{n+\frac{2}{3}}-3y_{n+\frac{1}{3}}\right)+\frac{h}{1080}\left(8f_n-9f_{n+\frac{1}{3}}+414f_{n+\frac{2}{3}}+127f_{n+1}\right) \tag{2.12}$$

Taylor's Series Algorithm for the Implementation of the Method

To generate y values for the approximate solution, the scheme and its first derivative are expanded term by term, up to the order of the scheme, by Taylor's series.

$$y_{n+i}=y(x_n+ih)=y_n+ihy'(x_n)+\frac{(ih)^2}{2!}f_n+.......... \tag{3.1}$$

$$y'_{n+i}=y'(x_n)+ihf_n+\frac{(ih)^2}{2!}f'_n+......... \tag{3.2}$$

and

$$f_{n+i}=y''(x_n+ih)=f_n+ihf'_n+\frac{(ih)^2}{2!}f''_n+...... \tag{3.3}$$

Where

$$f_n=f(x_n,y_n,y'_n)$$
$$f^{(i)}=f^{(i)}(x_n,y_n,y'_n),i=1,2,3$$

We find f', f'' and f''' by the use of partial derivative techniques as follows:

$$f'={df}/{dx}={\partial f}/{\partial x}+y'{\partial f}/{\partial x}+f{\partial f}/{\partial y'}$$
$$f''={d^2f}/{dx^2}=2(Ay'+Bf)+Cf_{y'}+D+E \tag{3.4}$$

Where

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$$A = \frac{\partial^2 f}{\partial x \partial y} + f \frac{\partial^2 f}{\partial y \partial y^1} \qquad B = \frac{\partial^2 f}{\partial x \partial y^1}$$

$$C = \frac{\partial f}{\partial x} + y^1 \frac{\partial f}{\partial y} + f \frac{\partial f}{\partial y^1}$$

$$D = \frac{\partial^2 f}{\partial x^2} + \left(y^1\right)^2 \frac{\partial^2 f}{\partial y^2} + f^2 \frac{\partial^2 f}{\left(\partial y^1\right)^2} \qquad E = f \frac{\partial f}{\partial y}$$

(3.5)

and

$$f''' = \frac{d^3 f}{dx^3} = 2G + 3\left(Hy' + If\right) + Jf_{y^1} + K + L + M$$

Where

$$G = y'f' \frac{\partial^2 f}{\partial y \partial y'} + f' \frac{\partial^2 f}{\left(\partial y'\right)^2} + y'ff_{y^1} \frac{\partial^2 f}{\partial y \partial y'} + f' \frac{\partial^2 f}{\partial x \partial y'}$$

$$H = \frac{\partial^3 f}{\partial x^2 \partial y} + y' \frac{\partial^3 f}{\partial x \partial y^2} + f \frac{\partial^2 f}{\partial y^2} + y'f \frac{\partial^3 f}{\partial y^2 \partial y'} + f^2 \frac{\partial^3 f}{\partial y\left(\partial y'\right)^2} + 2 \frac{\partial^3 f}{\partial x \partial y \partial y'}$$

$$I = \frac{\partial^3 f}{\partial x^2 \partial y'} + \frac{\partial^2 f}{\partial x \partial y} + f \frac{\partial^3 f}{\partial x\left(\partial y'\right)^2} + f \frac{\partial^2 f}{\partial y \partial y'}$$

$$J = f \frac{\partial f}{\partial y} + 2y' \frac{\partial^2 f}{\partial x \partial y} + \left(y'\right)^2 \frac{\partial^2 f}{\partial y^2} + f' \frac{\partial f}{\partial y^1} + 2f \frac{\partial^2 f}{\partial x \partial y'} + f^2 \frac{\partial^2 f}{\left(\partial y'\right)^2}$$

$$K = \left(\frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + f \frac{\partial f}{\partial y'}\right)\left[\frac{\partial^2 f}{\partial x \partial y'} + y' \frac{\partial^2 f}{\partial y \partial y'} + f \frac{\partial^2 f}{\left(\partial y'\right)^2}\right]$$

$$L = \frac{\partial^3 f}{\partial x^3} + f^3 \frac{\partial^3 f}{\left(\partial y'\right)^3} + \left(y'\right)^3 \frac{\partial^3 f}{\partial y^3}$$

$$M = f' \frac{\partial f}{\partial y}$$

Analysis of the Basic Properties of the Method

Order of accuracy and Error Constant

In line with Lambert (1973), the local truncation error associated with K – step linear multistep method, is taken to be linear difference operator:

$$L\left[y(x), h\right] = \sum_{j=0}^{k} \left\{\alpha_j y\left(x_{n+j}\right) - h\beta_j y\left(x_{n+j}\right)\right\} \qquad (4.1)$$

Equation (2.11) can be expanded as a Taylor's series about the point x if $y(x)$ is sufficiently differentiable to obtain the expression:

$$L\left[y(x), h\right] = C_0 y\left(x_n\right) + C_1 hy'\left(x_n\right) + C_2 h^2 y''\left(x_n\right) + \ldots + C_2 h^q y^q\left(x_n\right) + \ldots, \qquad (4.2)$$

where $C_{q,} \; q = 0,1,\ldots,$ are the constant coefficients given as:

$$C_0 = \sum_{j=0}^{k} \alpha_j , \quad (4.3)$$

$$C_1 = \sum_{j=0}^{k} j\alpha_j , \quad (4.4)$$

And $$C_q = \frac{1}{q!}\left[\sum_{j=0}^{k} j\alpha_j - q(q-1)\left(\sum_{j=0}^{k} j^{q-1}\beta_j + \sum_{j=0}^{k} v^{q-1}\beta_{vj}\right)\right], \quad (4.5)$$

In line with Lambert (1973), we say that the k – step, linear multistep method (2.11) has order p if $C_0 = C_1 = \ldots = C_{p-1} = C_p$ and $C_{p+1} \neq 0$. Thus C_{p+1} is the error constant of the method. From our calculation, subjecting our schemes to equations (4.2)-(4.5), it is established that our linear multistep schemes have high order of accuracy $p = 4$, and relatively small error constants $C_6 = \frac{-1}{174960}$.

Consistency

A linear multistep method is said to be consistent if:

(i) The order $p \geq 1$, (ii) $\sum_{j=0}^{k} \alpha_j = 0$,

(ii) $\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$, (iv) $\rho(1) = 0, \rho'(1) = \sigma(1)$,

Where ρ and σ are the first and second characteristic polynomials of our method (2.11), on applying these aforelisted definitions to our method it was found to be consistent.

Zero Stability of the method

A linear multistep method is said to be Zero stable if no roots of the first characteristic polynomial $\rho(r)$ has modulus greater than one, and if every root of the modulus one is simple (Lambert (1973). In the same way, by applying this definition to our method (2.11), it was found to be Zero stable.

Test Problems

We then translate the scheme (2.11), its derivative scheme (2.12) and the Taylor's series algorithm needed for its implementation into codes using FORTRAN Programming language and adopt to solve the following test problems to test the accuracy of the method:

Problem 1:

We consider an initial value problem:

$$y'' - x(y')^2 = 0 \quad y(0) = 1, y'(0) = \frac{1}{2} \quad \text{with } h = \frac{0.1}{32} \quad (4.1)$$

Whose exact solution is:

$$y = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right) \quad (4.2)$$

The Numerical result to the problem is as shown in the table 1 below, compared with the results obtained by Adesanya (2011) of order 6, implemented in block mode.

Problem 2:

We consider an initial Value problem solved with an existing method by Yahaya and Badmus (2009) of order 4 implemented in block mode:

$$y'' = y' \quad y(0) = 0, y'(0) = -1 \quad \text{with } h = 0.1 \quad (4.3)$$

Whose exact solution is:

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$$y(x) = 1 - \exp(x) \tag{4.4}$$

The numerical result is as shown in table II below.

Conclusion

In this research work, a new Linear Multistep Hybrid method implemented with Taylor's series algorithm for direct solutions of general second order initial value problems of ordinary differential equations is developed.

The use of Taylor's series algorithm for the implementation of the method has enabled derivatives of continuous schemes to any possible order to be computed. This allows direct solution of Initial Value Problems of ordinary differential equations without reduction to systems of first order differential equations.

The accuracy of the derived method is tested with two test problems and the results were compared with Adesanya (2011) of order 6 with step length k=5 and Yahaya and Badmus (2009) of order 4 with step length k=2, which were implemented in block mode. The new method gives better accuracy.

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Table I: Results and errors for problem (4.1)

x	YEX	YC	ERRold	ERRnew
0.2	1.100335347731	1.100335347729	5.4257e-010	1.5863e-012
0.4	1.202732554054	1.202732554047	4.7893e-009	6.6722e-012
0.6	1.309519604203	1.309519604185	1.8699e-008	1.7757e-011
0.8	1.423648930194	1.423648930153	5.4759e-008	4.0842e-011
1.0	1.549306144334	1.549306144243	1.4340e-008	9.1080e-011

Table II: Results and errors for problem (4.3)

X	YEX	YC	ERRold	ERRnew
0.2	-0.221402758160	-0.221402668007	0.3267e-003	9.0153e-008
0.4	-0.491824697641	-0.491824299828	0.4857e-002	3.9781e-007
0.6	-0.822118800391	-0.822117785516	0.1439e-001	1.0149e-006
0.8	-1.225540928493	-1.225538865231	0.2990e-001	2.0633e-006
1.0	-1.718281828459	-1.718278124741	0.5255e-001	3.7037e-006

Note: YEX= Y Exact, YC= YComputed, ERRold = Error in the old method,
ERRnew = Error in the new method.

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